W12. Prove that:
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$$\int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x} dx = \frac{n\pi}{2}, n \in \mathbb{N}.$$

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First note that $\int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x} dx = \int_0^{\pi/2} \frac{1-\cos 2nx}{1-\cos 2x} dx = \frac{1}{2} I_n$, where $I_n := \int_0^{\pi} \frac{1-\cos nx}{1-\cos x} dx$.

Let $c_n := \cos nx, t := \cos x, d_n := 1 - c_n \Rightarrow c_n = 1 - d_n$.

Then $\cos(n+1)x + \cos(n-1)x = 2\cos x \cdot \cos nx$ becomes

$c_{n+1} - 2t \cdot c_n + c_{n-1} = 0 \Leftrightarrow 1 - d_{n+1} - 2t(1-d_n) + 1 - d_{n-1} = 0 \Leftrightarrow$

$2td_n - 2t - d_{n-1} - d_{n+1} + 2 = 0 \Leftrightarrow d_{n+1} + d_{n-1} - 2td_n = 2(1-t) \Leftrightarrow$

$$\frac{1-\cos(n+1)x}{1-\cos x} + \frac{1-\cos(n-1)x}{1-\cos x} - 2\cos x \cdot \frac{1-\cos nx}{1-\cos x} = 2 \cdot \frac{1-\cos x}{1-\cos x}$$

and, therefore, $I_{n+1} + I_{n-1} - 2\int_0^{\pi} \cos x \cdot \frac{1-\cos nx}{1-\cos x} dx = 2\int_0^{\pi} dx = \pi$

Since $\int_0^{\pi} \cos x \cdot \frac{1-\cos nx}{1-\cos x} dx = \int_0^{\pi} \left((\cos x - 1 + 1) \cdot \frac{1-\cos nx}{1-\cos x} \right) dx =$

$\int_0^{\pi} (\cos nx - 1) dx + I_n$ and $\int_0^{\pi} (\cos nx - 1)\, dx = \int_0^{\pi} \cos(nx) dx - \pi =$

$\left(\frac{\sin nx}{n} \right)_0^{\pi} - \pi = -\pi$ then $I_{n+1} + I_{n-1} - 2(-\pi + I_n) = 2\pi \Leftrightarrow$

$$I_{n+1} + I_{n-1} - 2I_n = 0, n \in \mathbb{N}.$$

Noting that $I_0 = 0, I_1 = \pi$ we obtain $I_{n+1} + I_{n-1} - 2I_n = 0 \Leftrightarrow$

$I_{n+1} - I_n = I_n - I_{n-1} \Rightarrow I_{n+1} - I_n = \pi \Rightarrow I_n = n\pi$

and, therefore, $\int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x} dx = \frac{n\pi}{2}$.